Progress in Solving the 3-Dimensional

Inversion Problem for Eddy Current NDE

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ABSTRACT

The eddy current NDE inversion problem is to determine the parameters of a flaw from the measured eddy current sensor impedance changes. Mathematically, this requires finding the transformation which gives the sensor impedance changes in terms of the flaw parameters, and then inverting this transformation. Finding the transformation is called the forward problem, and finding the inverse of the transformation is equivalent to the inversion problem. The principal difficulty in solving the forward problem is finding solutions to Maxwell's equations in the complex geometries involved. This paper describes a solution to the forward problem which is valid for ellipsoidal shaped void flaws in a non-magnetic conductor, and for flaw dimensions such that the incident field variations are at most linear over the region occupied by the flaw.

INTRODUCTION

The eddy current NDE inversion problem is to determine the parameters of a flaw from the measured eddy current sensor impedance changes. Mathematically, this requires finding the transformation which gives the sensor impedance changes in terms of the flaw parameters, and then inverting this transformation. Finding the transformation is called the forward problem, and finding the inverse is equivalent to the inversion problem.

The principal difficulty in solving the forward problem is finding the solution to Maxwell's equations in the complex geometries involved. The first significant contribution to this problem was the work of Burrows [1], who used the reciprocity theorem and a scattering theory to derive the general form of the transformation between the sensor voltage change and the incident and scattered fields associated with the flaw. In order to get around the difficulties associated with solving Maxwell's equations, Burrows assumed the flaw to be small compared to the spatial variations of the incident field. This is usually not the case, and it eliminates important phase information. The next major advance was made by Dodd et al [2], who recognized that the field incident upon the flaw could be computed by numerical integration techniques, and then Burrows' result could be used to calculate the change in sensor impedance. However, the restriction to flaws which are small compared to the incident field variation remained.

In a previous paper [3] the authors made two advances upon these ideas. First, the incident field was computed by the finite element method (FEM), which is very accurate and can be applied generally. (In fact, the FEM is good enough to solve the forward problem numerically, although this is expensive since the computation must be made for each value of the flaw parameter, and it does not yield the "understanding" of a formula) Second, Burrows' scattering theory was extended to the case of a flaw which is small enough that the incident field varies at most linearly over the flaw dimensions. However, both the FEM and the scattering theory were developed only for two dimensional problems.

In this paper, the scattering theory is extended to three dimensions for the case of an ellipsoidal shaped void flaw in a non-magnetic conductor, with flaw dimensions such that the incident field variations are at most linear over the flaw dimensions.

THREE COMPONENT SCATTERING THEORY

The Scattering Model

In the scattering theory approach to the solution of the forward problem, the change in sensor impedance is found from the incident and scattered fields of the flaw by using the reciprocity theorem, as explained by Auld [4]. For a void flaw in a linear homogeneous, isotropic, conducting medium with free space permittivity and permeability, the change ΔZ in sensor impedance is given by

$$\Delta Z = \frac{1}{I^2} \int_{V_f} \sigma(\overline{E}.\overline{E}') \, dv \qquad (1)$$

where: σ = conductivity of the medium
$\underline{I}$ = the current at the sensor terminals
$\overline{E}'$ = the electric field without the flaw
$\overline{E}$ = the electric field with the flaw
dv = a differential volume element
V_f = the volume of the flaw

Therefore, to compute the sensor impedance change, it is necessary to compute the electric fields within the boundaries of the flaw both when the flaw is present and when it is not.

The strategy for computing these electric fields is to approximate the incident field in the vicinity of the flaw by its constant plus linearly varying components. The respective scattered fields are then approximated for an ellipsoidal flaw by the dipole and quadrupole field solutions to the static form of Maxwell's equations. For each component of the incident field, there will be a scattered field solution internal to the flaw and one external to the flaw.

The strengths of these scattered fields are found by matching the boundary conditions for the incident, exterior scattered, and interior scattered fields at the boundary of the ellipsoidal flaw. For a void flaw, the boundary conditions are:

1) the current normal to the flaw boundary is zero i.e. the electric field outside the flaw and normal to the boundary is zero

2) the electric field tangential to the flaw boundary is continuous across the boundary

Note that condition 1) does not require the electric field inside the flaw and normal to the boundary to be zero. This field can terminate on charges on the surface of the flaw.

For the constant and linearly varying components of the incident field, condition 1) will be met respectively by the normal component of the external dipole and quadrupole scattered fields. Condition 2) is met by continuing the incident fields into the flaw to match the tangential components of the incident fields, and by adding internal dipole and quadrupole scattered fields to match the tangential components of the respective external scattered fields.

The Incident Field

The incident field is conveniently described in a cartesian coordinate system with origin at the center of the ellipsoidal flaw. The coordinates x, y, z are aligned with the principal axes of the ellipsoid. The general expression for the incident electric field induced by the eddy current is

$$E = \bar{i}_x E_x + \bar{i}_y E_y + \bar{i}_z E_z \tag{2}$$

where $\bar{i}_z$ is a unit vector in the x-direction, and similarly for $\bar{i}_y$, $\bar{i}_z$. Expanding each of the field components in a Taylor series about the origin gives, to first order

$$\begin{aligned} E_x &= E_1 + A_{11}x + A_{12}y + A_{13}z \\ E_y &= E_2 + A_{21}x + A_{22}y + A_{23}z \\ E_z &= E_3 + A_{31}x + A_{32}y + A_{33}z \end{aligned} \tag{3}$$

This approximation expresses the incident field as the sum of a constant plus a linearly varying field.

The field described by (2) and (3) must have zero divergence since it is induced by an electric current in a conductor. For this to be true, the trace of the matrix of coefficients A_{ij} must be zero. In many cases of practical interest this condition is satisfied because the diagonal terms are individually zero.

The curl of the field described by (2) and (3) will be zero if and only if the matrix of coefficients A_{ij} is symmetric. In general, this will not be true. Thus the incident field cannot be uniquely described as the gradient of a scalar potential. Because of this, the scattering problem is solved here directly in terms of the electric fields rather than potentials.

The Scattered Fields

The scattered fields are most conveniently described in an ellipsoidal coordinate system. The particular form described by Stratton [5] is used here. In this system, the coordinates ξ, η, and ζ are defined by their transformation to cartesian coordinates

$$\begin{aligned} x &= \pm \frac{(\xi + a^2)(\eta + a^2)(\zeta + a^2)}{(b^2 - a^2)(c^2 - a^2)}, \\ y &= \pm \frac{(\xi + b^2)(\eta + b^2)(\zeta + b^2)}{(c^2 - b^2)(a^2 - b^2)}, \\ z &= \pm \frac{(\xi + c^2)(\eta + c^2)(\zeta + c^2)}{(a^2 - c^2)(b^2 - c^2)}. \end{aligned} \tag{4}$$

where $-a^2 < \zeta < -b^2 < \eta < -c^2 < \xi$. The surface ξ = constant is an ellipsoid with principal axes of length a, b, c aligned along the x, y, z axes respectively. The surface η = constant is a hyperboloid of one sheet, and the surface ζ = constant a hyperboloid of two sheets. Since Maxwell's equations are linear in this application, the scattered fields can be determined separately for each of the 12 components of (3), and the results added to give the total scattered field.

The internal and external scattered fields are approximated by solutions to the static form of Maxwell's equations. These solutions can be derived from potentials which are known solutions to Laplace's equations. In ellipsoidal coordinates, the solutions to Laplace's equation are the Lame functions [6].

For the components of the constant incident field, the potentials of the induced internal scattered fields are products of the Lame functions of the 1st kind of degree 1. These are

$$\begin{aligned} F_a(\lambda) &= \sqrt{\lambda + a^2} \\ F_b(\lambda) &= \sqrt{\lambda + b^2} \\ F_c(\lambda) &= \sqrt{\lambda + c^2} \end{aligned} \tag{5}$$

For example, for the $\bar{i}_x E_1$ component of the constant incident field, the potential of the internal scattered field is of the form

$$\psi_{xi} = K_{xi} F_a(\xi) F_a(\eta) F_a(\zeta) \quad . \tag{6}$$

The field derived from this potential is called an "internal dipole" field in this paper. The field is actually constant.

The potentials of the corresponding external scattered fields are products of the Lame functions of the 1st and 2nd kind of degree 1. The Lame functions of the 2nd kind of degree 1 are the of the form

$$G_a(\lambda) = H_a(\lambda) F_a(\lambda) \tag{7}$$

$$\text{where } H_a(\lambda) = \int_\lambda^\infty \frac{ds}{R(s)\, F_a^2(s)} \tag{8}$$

$$\text{and } R(s) = \sqrt{(s + a^2)(s = b^2)(s + c^2)} \tag{9}$$

with similar expressions for $G_b(\lambda)$ and $G_c(\lambda)$. As an example, the potential of the external scattered field induced by the $\bar{i}_x E_1$ component

$$\psi_{xe} = K_{xe} G_a(\xi) F_z(\eta) F_z(\zeta) \quad . \tag{10}$$

The field derived from this potential is a true dipole field, and is called an "external dipole" field in this paper.

The total electric field solution for a field component $\bar{i}_x E_1$ incident upon an ellipsoidal void is found by matching the boundary conditions 1) and 2) above at the surface of the flaw.

The procedure is as follows. The amplitude of the external dipole field is determined by setting its ξ component equal and opposite to the ξ component of the constant incident field component at the flaw boundary, $\xi = 0$. This satisfies condition 1), i.e. the current normal to the flaw boundary is zero. Next, the constant incident field is continued into the flaw interior to satisfy condition 2) for the constant incident field, i.e. the tangential electric field is continuous across the boundary. Finally, the η and ζ components of the internal dipole field are made equal in amplitude to the η and ζ components of the external dipole field at the flaw boundary, $\xi = 0$. This satisfies condition 2) for the external dipole. The ξ components of the internal fields are terminated by charges on the conducting surface of the flaw. By applying this procedure for each of the 3 cartesian components of the constant part of the incident field, the amplitudes of the induced internal fields can be found. It is these internal fields which are needed for the $\bar{E}'$ term in (1), along with the internal fields induced by the components of the linearly varying part of the incident field.

The total internal fields induced by the constant field components are given in Table 1, along with their special forms for the case of a sphere (a=b=c) and a disc (a=b, c=o) flaw. Note that the total internal field for each component is the sum of two fields, a constant field equal to the incident field, plus an internal dipole field (which is also a constant field).

For the components of the linearly varying incident field, the potentials of the induced internal scattered fields are products of the Lame functions of the first kind of degree 2. Only components of the linearly varying incident field corresponding to the off-diagonal terms in (3) will be considered here, since the diagonal terms are usually zero in practical problems of interest.

For this case, the corresponding Lame functions of the first kind of degree 2 are

$$\begin{aligned} F_{ab}(\lambda) &= \sqrt{(\lambda + a^2)(\lambda + b^2)} \\ F_{bc}(\lambda) &= \sqrt{(\lambda + b^2)(\lambda + c^2)} \\ F_{ca}(\lambda) &= \sqrt{(\lambda + c^2)(\lambda + a^2)} \quad . \end{aligned} \tag{11}$$

For example, for the $\bar{i}_x A_{12} y$ component of the incident field, the potential of the internal scattered field is of the form

$$\psi_{xyi} = K_{xyi} F_{ab}(\xi) F_{ab}(\eta) F_{ab}(\zeta) \quad . \tag{12}$$

Analogous to the case of the dipole scattered field above, it is convenient to call the field derived from this potential an "internal quadrupole" field.

The potentials of the corresponding "external quadrupole" scattered fields are products of the Lame functions of the 2nd kind of degree 1 and degree 2. The Lame functions of the 2nd kind of degree 2 are of the form

$$G_{ab}(\lambda) = H_{ab}(\lambda) F_{ab}(\lambda) \tag{13}$$

$$\text{where } H_{ab}(\lambda) = \int_\lambda^\infty \frac{ds}{R(s)\, F_{ab}^2(S)} \tag{14}$$

with similar expressions for $G_{bc}(\lambda)$ and $G_{ca}(\lambda)$. As an example, the potential of the external

Table 1 - Internal Scattered Fields for a Constant Incident Field

INCIDENT	INTERNAL: Ellipsoid	Sphere	Disc
$\bar{i}_x E_1$	$\bar{i}_x E_1 - \bar{i}_x \frac{1}{2a^2} \frac{H_a(0)}{P_b(0)} E_1$	$\bar{i}_x \frac{3}{2} E_1$	$\bar{i}_x E_1$
$\bar{i}_y E_2$	$\bar{i}_y E_2 - \bar{i}_y \frac{1}{2b^2} \frac{H_b(0)}{P_b(0)} E_2$	$\bar{i}_y \frac{3}{2} E_2$	$\bar{i}_y E_2$
$\bar{i}_z E_3$	$\bar{i}_z E_3 - \bar{i}_z \frac{1}{2c^2} \frac{H_c(0)}{P_c(0)} E_3$	$\bar{i}_z \frac{3}{2} E_3$	$\bar{i}_z \frac{2}{\pi} \frac{a}{c} E_3$

where $P_a(0) = \frac{1}{2a^2} H_a(0) + H_a'(0)$

Table 2 - Internal Scattered Fields for a Linearly Varying Incident Field

INCIDENT	INTERNAL: Ellipsoid	Sphere	Disc
$\bar{i}_x A_{12}y$	$\bar{i}_x A_{12}y - \frac{1}{2a^2} \frac{H_{ab}(0)}{P_{ab}(0)} A_{12} \quad (\bar{i}_x y + \bar{i}_y x)$	$\bar{i}_x \frac{4}{3} A_{12}y + \bar{i}_y \frac{1}{3} A_{12}x$	$\bar{i}_x A_{12}y$
$\bar{i}_x A_{13}z$	$\bar{i}_x A_{13}z - \frac{1}{2a^2} \frac{H_{ac}(0)}{P_{ac}(0)} A_{13} \quad (\bar{i}_x z + \bar{i}_z x)$	$\bar{i}_x \frac{4}{3} A_{13}z + \bar{i}_z \frac{1}{3} A_{13}x$	$\bar{i}_x A_{13}z$
$\bar{i}_y A_{21}x$	$\bar{i}_y A_{21}x - \frac{1}{2b^2} \frac{H_{ba}(0)}{P_{ba}(0)} A_{21} \quad (\bar{i}_y x + \bar{i}_x y)$	$\bar{i}_y \frac{4}{3} A_{21}x + \bar{i}_y \frac{1}{3} A_{21}y$	$\bar{i}_y A_{21}x$
$\bar{i}_y A_{23}z$	$\bar{i}_y A_{23}z - \frac{1}{2b^2} \frac{H_{bc}(0)}{P_{bc}(0)} A_{23} \quad (\bar{i}_y z + \bar{i}_z y)$	$\bar{i}_y \frac{4}{3} A_{23}z + \bar{i}_z \frac{1}{3} A_{23}y$	$\bar{i}_y A_{23}z$
$\bar{i}_z A_{31}x$	$\bar{i}_z A_{31}x - \frac{1}{2c^2} \frac{H_{ca}(0)}{P_{ca}(0)} A_{31} \quad (\bar{i}_z x + \bar{i}_x z)$	$\bar{i}_z \frac{4}{3} A_{31}x + \bar{i}_x \frac{1}{3} A_{31}z$	$\frac{4}{3\pi} \frac{a}{c} (\bar{i}_z A_{31}x + \bar{i}_x A_{31}z)$
$\bar{i}_z A_{32}y$	$\bar{i}_z A_{32}y - \frac{1}{2c^2} \frac{H_{cb}(0)}{P_{cb}(0)} A_{32} \quad (\bar{i}_z y + \bar{i}_y z)$	$\bar{i}_z \frac{4}{3} A_{32}y + \bar{i}_y \frac{1}{3} A_{32}z$	$\frac{4}{3\pi} \frac{a}{c} (\bar{i}_z A_{32}y + \bar{i}_y A_{32}x)$

where $P_{ab}(0) = \frac{1}{2} (\frac{1}{a^2} + \frac{1}{b^2}) H_{ab}(0) + H'_{ab}(0)$

quadrupole scattered field induced by the $i_x A_{12} y$ component of the incident field is

$$\psi_{xye} = K_{xye}\, G_{ab}(\xi)\, F_{ab}(\eta)\, F_{ab}(\zeta) \qquad (15)$$

The total electric field solution for a field component $\bar{i}_x A_{12} y$ incident upon an ellipsoidal void is found by matching the boundary conditions in a manner similar to the procedure described above for the constant incident fields and dipole scattered fields.

The total internal fields induced by the linearly varying field components are given in Table 2, along with their special forms for the case of a sphere and a disc flaw. Analogous to the results shown in Table 1, note that the total internal field for each component is the sum of two components, a linearly varying field equal to this incident field, plus an internal quadrupole field (which is the sum of two linearly varying fields).

EXAMPLES

Spherical Flaw

Consider the case of a spherical void flaw of radius a in the volume of a conducting material with conductivity σ. Assume the incident field is only in the x-direction, and that it can be represented in the region of the flaw by the approximation

$$\bar{E} = \bar{i}_x (E_1 + A_{12} y) \qquad (16)$$

where E_1 and A_{12} are complex constants.

The field internal to the flaw is the sum of the responses to the constant and linearly varying fields respectively, as given in Tables 1 and 2.

$$\bar{E}' = \bar{i}_x \frac{3}{2} E_1 + \bar{i}_x \frac{4}{3} A_{12} y + \bar{i}_y \frac{1}{3} A_{12} x \qquad (17)$$

substituting the fields $\bar{E}$ and $\bar{E}'$ into (1) and integrating over the volume of the flaw gives

$$\Delta Z = \frac{\sigma}{I^2} \left(2\pi a^3 E_1^{\,2} + \frac{16}{45} \pi a^5 A_{12}^2\right). \qquad (18)$$

Disc Shaped Flaw

Consider the case of a disc shaped flaw, which is an ellipsoid with a = b and $c \to 0$, in the volume of a conducting material of conductivity σ. The principal axes a, b, c are aligned with the coordinates x, y, z respectively.

Assume first that the incident field is given by (16) above. Then the field internal to the flaw is the sum of the responses to the constant and linearly varying fields respectively, as given in Tables 1 and 2.

$$\bar{E} = \bar{i}_x (E_1 + A_{12} y) \qquad (19)$$

as $c \to 0$, the volume of the flaw goes to zero, and the integral in (1) goes to zero. Therefore,

$$\Delta Z = 0 \qquad (20)$$

which says that a crack cannot be detected by a field which sees only its edge.

Assume now that the incident field is given by

$$\bar{E} = \bar{i}_z (E_3 + A_{31} x). \qquad (21)$$

For this case, the field internal to the flaw is found from Tables 1 and 2 to be

$$\bar{E}' = \bar{i}_z \frac{2}{\pi} \frac{a}{c} E_3 + \left(\frac{4}{3\pi} \frac{a}{c} A_{31}\right) (\bar{i}_z x + \bar{i}_x z). \qquad (22)$$

The corresponding impedance change is found from (1) to be

$$\Delta Z = \frac{\sigma}{I^2} \left(\frac{8}{3} a^3 E_3^{\,2} + \frac{16}{45} a^5 A_{31}^{2}\right). \qquad (23)$$

Note that ΔZ is independent of the thickness c of the flaw when $c \to 0$.

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